St Petersburg University Preprint SPbU-IP-98-10; *chao-dyn/9806004*

Renormalization Group and Anomalous Scaling in a Simple Model of Passive Scalar Advection in Compressible Flow

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Abstract

Field theoretical renormalization group methods are applied to a simple model of a passive scalar quantity advected by the Gaussian non-solenoidal ("compressible") velocity field with the covariance $\propto \delta(t-t')|\mathbf{x}-\mathbf{x}'|^{\varepsilon}$. Convective range anomalous scaling for the structure functions and various pair correlators is established, and the corresponding anomalous exponents are calculated to the order ε^2 of the ε expansion. These exponents are non-universal, as a result of the degeneracy of the RG fixed point. In contrast to the case of a purely solenoidal velocity field (Obukhov–Kraichnan model), the correlation functions in the case at hand exhibit nontrivial dependence on both the IR and UV characteristic scales, and the anomalous scaling appears already at the level of the pair correlator. The powers of the scalar field *without derivatives*, whose critical dimensions determine the anomalous exponents, exhibit multifractal behavior. The exact solution for the pair correlator is obtained; it is in agreement with the result obtained within the ε expansion. The anomalous exponents for passively advected magnetic fields are also presented in the first order of the ε expansion.

PACS number(s): 47.10.+g, 47.25.Cg, 05.40.+j

I. INTRODUCTION

Much attention has been paid recently to a simple model of the passive advection of a scalar quantity by a Gaussian short-correlated velocity field, introduced by Obukhov [1] and Kraichnan [2], see the papers [3–24] and references therein. The structure functions of the scalar field in this model exhibit anomalous scaling behavior, and the corresponding anomalous exponents can be calculated explicitly using certain physically motivated "linear ansatz" [3], within regular expansions in various small parameters [5–10,16,22], and using numerical simulations [4,18,21,23]. On the other hand, this model provides a good testing ground for various concepts and methods of the turbulence theory: closure approximations [3,4,15], refined similarity relations [12–14], Monte Carlo simulations [15,23], renormalization group [22], and so on.

The advection of a passive scalar field $\theta(x) \equiv \theta(t, \mathbf{x})$ is described by the stochastic equation

$$\partial_t \theta + \partial_i (v_i \theta) = \nu_0 \triangle \theta + f, \tag{1.1}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, ν_0 is the molecular diffusivity coefficient, $\triangle$ is the Laplace operator, $\mathbf{v}(x)$ is the transverse (owing to the incompressibility) velocity field, and $f \equiv f(x)$ is a Gaussian scalar noise with zero mean and correlator

$$\langle f(x) f(x') \rangle = \delta(t - t')\, C(Mr), \quad r \equiv |\mathbf{x} - \mathbf{x}'|. \tag{1.2}$$

The parameter $L \equiv M^{-1}$ is an integral scale related to the scalar noise, and $C(Mr)$ is some function finite as $L \to \infty$. Without loss of generality, we take $C(0) = 1$ (the dimensional coefficient in (1.2) can be absorbed by appropriate rescaling of the field θ and noise f).

In the real problem, the field $\mathbf{v}(x)$ satisfies the Navier–Stokes equation. In the simplified model considered in [2–8], $\mathbf{v}(x)$ obeys a Gaussian distribution with zero average and correlator

$$\langle v_i(x) v_j(x') \rangle = D_0 \frac{\delta(t - t')}{(2\pi)^d} \int d\mathbf{k}\, P_{ij}(\mathbf{k})\, (k^2 + m^2)^{-d/2-\varepsilon/2} \exp[\mathrm{i}\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')], \tag{1.3}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector, $k \equiv |\mathbf{k}|$, $D_0 > 0$ is an amplitude factor, $1/m$ is another integral scale, and d is the dimensionality of the $\mathbf{x}$ space; $0 < \varepsilon < 2$ is a parameter with the real ("Kolmogorov") value $\varepsilon = 4/3$. The relations

$$D_0/\nu_0 \equiv g_0 \equiv \Lambda^{\varepsilon} \tag{1.4}$$

define the coupling constant ("charge") g_0 and the characteristic ultraviolet (UV) momentum scale Λ.

The quantities of interest are, in particular, the single-time structure functions

$$S_n(r) \equiv \langle [\theta(t, \mathbf{x}) - \theta(t, \mathbf{x}')]^n \rangle, \quad r \equiv |\mathbf{x} - \mathbf{x}'|. \tag{1.5}$$

In the model (1.1)–(1.3), the odd multipoint correlation functions of the scalar field vanish, while the even single-time functions satisfy linear partial differential equations [2], see also [5,7,24]. The solution for the pair correlator is obtained explicitly; it shows that the structure

function S_2 is finite for M, $m = 0$ [2]. The higher-order correlators are not found explicitly, but their asymptotic behavior for $M \to 0$ can be extracted from the analysis of the nontrivial zero modes of the corresponding differential operators in the limits $1/d \to 0$ [5,6], $\varepsilon \to 0$ [7,8], or $\varepsilon \to 2$ [9,16]. It was shown that the structure functions are finite for $m = 0$, and in the convective range $\Lambda >> 1/r >> M$ they have the form (up to the notation)

$$S_{2n}(r) \propto D_0^{-n}\, r^{n(2-\varepsilon)}\, (Mr)^{\Delta_n}, \tag{1.6}$$

with negative anomalous exponents Δ_n, whose first terms of the expansion in $1/d$ [5,6] and ε [7,8] have the form

$$\Delta_n = -2n(n-1)\varepsilon/(d+2) + O(\varepsilon^2) = -2n(n-1)\varepsilon/d + O(1/d^2). \tag{1.7}$$

In the paper [22], the field theoretical renormalization group (RG) and operator product expansion (OPE) were applied to the model (1.1)–(1.3). In the RG approach, the anomalous scaling for the structure functions and various pair correlators is established as a consequence of the existence in the corresponding operator product expansions of "dangerous" composite operators [powers of the local dissipation rate], whose *negative* critical dimensions determine the anomalous exponents Δ_n. The exponent ε plays in the RG approach the role analogous to that played by the parameter $\varepsilon = 4 - d$ in the RG theory of critical phenomena [25]. The anomalous exponents were calculated in [22] to the order ε^2 of the ε expansion for the arbitrary value of d, and they are in agreement with the first-order results obtained in the zero-modes approach [5–8]. The RG approach to the stochastic theory of turbulence is reviewed in [26].

In the paper [2], a closure-type approximation for the model (1.1)–(1.3), the so-called linear ansatz, was used to derive simple explicit expression for the anomalous exponents for any $0 < \varepsilon < 2$, d, and n. Although the predictions of the linear ansatz appear consistent with some numerical simulations [3,23] and exact relations [11,19], they do not agree with the results obtained within the zero-modes and RG approaches in the ranges of small ε, $2-\varepsilon$ or $1/d$. This disagreement can be related to the fact that these limits have strongly nonlocal dynamics in the momentum space, which suggests possible relation between deviations from the linear ansatz and locality of the interactions, see discussion in Refs. [19,23]. [1]

The results of the RG approach are completely reliable and internally consistent for small ε, but the validity of their extrapolation to the finite values of ε is not obvious. Most numerical simulations have been limited to two dimensions [4,21] and have not yet been able to cover the small ε or large d ranges, in which the reliable analytical results are available. Therefore, it is not yet clear whether the anomalous scaling in the small ε and finite ε ranges has the same origin, with the exponents depending continuously on ε, or there is a "crossover" in the anomalous scaling behavior for some small but finite value of ε and these ranges should be treated separately.

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[1] The small ε limit can be treated perturbatively, the effective small parameter equals to the reciprocal of the significant range of interactions in the momentum space. This range becomes infinite as ε goes to zero [27].

Another important question is that of the universality of anomalous exponents. The exponents Δ_n in (1.7) do not depend on the choice of the correlator (1.2) and on the specific form of the infrared (IR) regularization in the correlator (1.3). It was argued on phenomenological grounds in [9] that the anomalous exponents in the Gaussian model can depend on more details of the velocity statistics than the exponent ε. The exponents indeed change when the function $\delta(t-t')$ in the correlator (1.3) is replaced by some function with finite width, i.e., the velocity has short but finite correlation time [10], and when the velocity field is taken to be time-independent (see Sec. V of the Ref. [22]).

In this paper, we consider the generalization of the model (1.1)–(1.3) to the case of a non-solenoidal ("compressible") velocity field. In this case, the correlator (1.3) is replaced by

$$\langle v_i(x)v_j(x')\rangle = \frac{\delta(t-t')}{(2\pi)^d}\int d\mathbf{k}\,\frac{D_0\,P_{ij}(\mathbf{k})+D_0'\,Q_{ij}(\mathbf{k})}{(k^2+m^2)^{d/2+\varepsilon/2}}\,\exp[\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')]. \tag{1.8}$$

The notation is explained below the Eq. (1.3); the new quantities are the longitudinal projector $Q_{ij}(\mathbf{k}) = k_ik_j/k^2$ and the additional amplitude factor $D_0' > 0$.

One should not expect that a Gaussian, white-noise model of the type (1.1), (1.2), (1.8) provides very good approximation for the real compressible advection; however, it can be used to illustrate the important distinctions which exist between the compressible and incompressible cases, see e.g. [28,29] and references therein.

The aim of this paper is to give the RG treatment of anomalous scaling with non-universal exponents; to compare the results of the ε expansion with the nontrivial exact exponent, and to present analytic results which probably will be easier to compare with numerical simulations than the analogous results for the incompressible case. We apply the RG method to the model (1.1), (1.2), (1.8) to establish the existence of the anomalous scaling in the convective range and to calculate the corresponding anomalous exponents to the second order of the ε expansion. We show that the single-time two-point correlation functions of the powers of the scalar field in the convective range have the form

$$\langle\theta^n(t,\mathbf{x})\theta^p(t,\mathbf{x}')\rangle \propto \nu_0^{-(n+p)/2}\,\Lambda^{-(n+p)}\,(\Lambda r)^{-\Delta_n-\Delta_p}(Mr)^{\Delta_{n+p}}, \quad r\equiv|\mathbf{x}-\mathbf{x}'|. \tag{1.9}$$

for even $n+p$ and zero otherwise. In addition to ε and d, the exponents Δ_n depend on a free parameter, the ratio $\alpha \equiv D_0'/D_0$ of the amplitudes in the correlator (1.8). In the first order of the expansion in ε they have the form

$$\Delta_n = n(-1+\varepsilon/2) - \frac{\alpha n(n-1)d\varepsilon}{2(d-1+\alpha)} + O(\varepsilon^2) \tag{1.10}$$

(the results $\Delta_1 = -1+\varepsilon/2$ for any α and $\Delta_n = n(-1+\varepsilon/2)$ for $\alpha = 0$ are in fact exact). We have also calculated the ε^2 term of the exponent Δ_n for any d and α; the result is rather cumbersome and is given in Sec. III.

The leading term of the convective range behavior of the structure functions (1.5) in the model (1.8) is completely determined by the contribution $\langle\theta^{2n}\rangle$; it is obtained from (1.9) by the substitution $n\to 2n$, $p\to 0$ and has the form

$$S_{2n}(r) \propto \nu_0^{-n}\,\Lambda^{-2n}\,(M/\Lambda)^{\Delta_{2n}}. \tag{1.11}$$

It follows from (1.9) that the anomalous scaling in the model (1.8) appears already at the level of the pair correlation function. The corresponding exponent Δ_2 is found exactly for all $0 < \varepsilon < 2$ from the exact solution for the single-time pair correlator, see Sec. II:

$$\Delta_2 = -2 - \frac{\varepsilon\,(\alpha - 1)\,(d-1)}{(d-1) + \alpha\,(1+\varepsilon)} \tag{1.12}$$

(anomalous scaling for the pair correlator with the exactly known exponent was established previously in [30] on the example of a passively advected magnetic field).

In the language of the RG, the non-universality of the exponents (1.10), (1.12) is explained by the fact that the fixed point of the RG equations is degenerated: its coordinate depends continuously on the ratio α (see Sec. III).

In contradistinction with the model (1.3), where the anomalous exponents are related to the critical dimensions of the composite operators $(\partial_i\theta\partial_i\theta)^n$ [22], the exponents Δ_n in (1.9), (1.11) are determined by the critical dimensions of the monomials θ^n, the powers of the field itself, and these dimensions appear to be nonlinear functions of n, see Sec. IV. This explains the difference between the convective range behavior of the model (1.3) and that of the model (1.8) and makes the limit $D'_0 \to 0$ rather subtle.

The model (1.8) remains nontrivial in the case $d = 1$, where the velocity field becomes purely potential. One can hope that the one-dimensional case is more accessible to study using numerical simulations, than the lowest-dimensional case $d = 2$ for (1.3), and it will be possible to compare the analytic results (1.9)–(1.12) with the numerical estimates (despite the fact that the structure functions (1.11) are independent of r, the values of the anomalous exponents can be extracted from their dependence on M). In the paper [20], the model (1.8) has been studied directly for the one-dimensional case in terms of certain potential function for the field θ; the analytic expressions for the anomalous exponents obtained within the zero-modes technique have been found to agree with non-perturbative numerical results. The relationship between our results and the results of [20] is discussed in Sec. II.

The paper is organized as follows. In Sec. II, we give the field theoretical formulation of the model (1.1), (1.2), (1.8) and derive exact equations for the response function and pair correlator of the scalar field. The explicit solution for the pair correlator is obtained and the exact expression (1.12) for the corresponding anomalous exponent is derived. In Sec. III, we perform the UV renormalization of the model and derive the corresponding RG equations with exactly known RG functions (the β function and the anomalous dimension). These equations have an IR stable fixed point, which establishes the existence of IR scaling with exactly known critical dimensions of the basic fields and parameters of the model. The solution of the RG equations for the correlation functions (1.9) is given, which determines their dependence on the UV scale. In Sec. IV, the dependence of the correlators on the IR scale is studied using the OPE, and the relations (1.9), (1.11) are derived. We also discuss briefly the RG approach to the model of passively advected magnetic fields introduced in [30]. In Sec. V, we present the calculation of the anomalous exponents in the model (1.8) to the order ε^2 of the ε expansion. The results obtained are briefly discussed in Sec. VI.

II. EXACT SOLUTION FOR THE PAIR CORRELATION FUNCTION

The single-time correlation functions of the field θ in the models of the type (1.1), (1.2), (1.3) or (1.8) satisfy closed linear partial differential equations [2] (see also Refs. [5,7,24]). Below we give an alternative derivation of the equation for the pair correlation functions based on the field theoretical formulation of the problem.

The stochastic problem (1.1), (1.2), (1.8) is equivalent to the field theoretical model of the set of three fields $\Phi \equiv \{\theta, \theta', \mathbf{v}\}$ with action functional

$$S(\Phi) = \theta' D_\theta \theta'/2 + \theta' \left[-\partial_t \theta - \boldsymbol{\partial}(\mathbf{v}\theta) + \nu_0 \triangle \theta\right] - \mathbf{v} D_v^{-1} \mathbf{v}/2. \tag{2.1}$$

The first four terms in (2.1) represent the Martin–Siggia–Rose-type action [31–34] for the stochastic problem (1.1), (1.2) at fixed $\mathbf{v}$, and the last term represents the Gaussian averaging over $\mathbf{v}$. Here D_θ and D_v are the correlators (1.2) and (1.8), respectively, the required integrations over $x = (t, \mathbf{x})$ and summations over the vector indices are understood.

The formulation (2.1) means that statistical averages of random quantities in stochastic problem (1.1), (1.2), (1.8) coincide with functional averages with the weight $\exp S(\Phi)$, so that the generating functionals of total ($G(A)$) and connected ($W(A)$) Green functions of the problem are represented by the functional integral

$$G(A) = \exp W(A) = \int \mathcal{D}\Phi \exp[S(\Phi) + A\Phi] \tag{2.2}$$

with arbitrary sources $A \equiv A^\theta, A^{\theta'}, A^{\mathbf{v}}$ in the linear form

$$A\Phi \equiv \int dx [A^\theta(x)\theta(x) + A^{\theta'}(x)\theta'(x) + A_i^{\mathbf{v}}(x) v_i(x)].$$

The model (2.1) corresponds to a standard Feynman diagrammatic technique with the triple vertex $-\theta' \boldsymbol{\partial}(\mathbf{v}\theta) \equiv \theta' V_j v_j \theta$ with vertex factor (in the momentum-frequency representation)

$$V_j = \mathrm{i} k_j, \tag{2.3}$$

where $\mathbf{k}$ is the momentum flowing into the vertex via the field θ'. The bare propagators in the momentum-frequency representation have the form

$$\langle \theta \theta' \rangle_0 = \langle \theta' \theta \rangle_0^* = (-i\omega + \nu_0 k^2)^{-1}, \tag{2.4a}$$

$$\langle \theta \theta \rangle_0 = C(k)\, (\omega^2 + \nu_0^2 k^4)^{-1}, \tag{2.4b}$$

$$\langle \theta' \theta' \rangle_0 = 0, \tag{2.4c}$$

where $C(k)$ is the Fourier transform of the function $C(Mr)$ from (1.2) and the bare propagator $\langle \mathbf{v}\mathbf{v} \rangle_0$ is given by Eq. (1.8). The parameter $g_0 \equiv D_0/\nu_0$ plays the part of the coupling constant in the perturbation theory. The pair correlation functions $\langle \Phi\Phi \rangle$ of the multicomponent field Φ satisfy standard Dyson equation, which in the component notation reduces to the system of two equations, cf. [35]

$$G^{-1}(\omega, k) = -\mathrm{i}\omega + \nu_0 k^2 - \Sigma_{\theta'\theta}(\omega, k), \tag{2.5a}$$

$$D(\omega, k) = |G(\omega, k)|^2 \, [C(k) + \Sigma_{\theta'\theta'}(\omega, k)], \tag{2.5b}$$

where $G(\omega, k) \equiv \langle \theta\theta' \rangle$ and $D(\omega, k) \equiv \langle \theta\theta \rangle$ are the exact response function and pair correlator, respectively, and $\Sigma_{\theta'\theta}$, $\Sigma_{\theta'\theta'}$ are self-energy operators represented by the corresponding 1-irreducible diagrams; the functions $\Sigma_{\theta\theta}$, $\Sigma_{\mathbf{vv}}$ in the model (2.1) vanish identically.

The feature characteristic of the models like (2.1) is that all the skeleton multiloop diagrams entering into the self-energy operators $\Sigma_{\theta'\theta}$, $\Sigma_{\theta'\theta'}$ contain effectively closed circuits of retarded propagators $\langle \theta\theta' \rangle$ and therefore vanish (it is also crucial here that the propagator $\langle \mathbf{vv} \rangle_0$ in (1.8) is proportional to the δ function in time). Therefore the self-energy operators in (2.5) are given by the single-loop approximation exactly and have the form[2]

$$\Sigma_{\theta'\theta}(\omega, k) = -\int \frac{d\omega'}{2\pi} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{D_0(k^2 - (\mathbf{k}\cdot\mathbf{q})^2/q^2) + D_0'(\mathbf{k}\cdot\mathbf{q})^2/q^2}{(q^2 + m^2)^{d/2+\varepsilon/2}} \, G(q', \omega'), \tag{2.6a}$$

$$\Sigma_{\theta'\theta'}(\omega, k) = \int \frac{d\omega'}{2\pi} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{D_0(k^2 - (\mathbf{k}\cdot\mathbf{q})^2/q^2) + D_0'(\mathbf{k}\cdot\mathbf{q})^2/q^2}{(q^2 + m^2)^{d/2+\varepsilon/2}} \, D(q', \omega'), \tag{2.6b}$$

where $q' \equiv |\mathbf{k} - \mathbf{q}|$. The integrations over ω' in the right hand sides of Eqs. (2.6) give the single-time response function $G(q) = (1/2\pi) \int d\omega' \, G(q, \omega')$ and the single-time pair correlator $D(q) = (1/2\pi) \int d\omega' \, D(q, \omega')$ (note that the both self-energy operators are in fact independent of ω). The only contribution to $G(q)$ comes from the bare propagator (2.4a), which in the t representation is discontinuous at coincident times. Since the correlator (1.8), which enters into the single-loop diagram for $\Sigma_{\theta'\theta}$, is symmetric in t and t', the response function must be defined at $t = t'$ by half the sum of the limits. This is equivalent to the convention $G(q) = (1/2\pi) \int d\omega' \, (-i\omega' + \nu_0 k^2)^{-1} = 1/2$ and gives

$$\Sigma_{\theta'\theta}(\omega, k) = (-1/2) \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{D_0(k^2 - (\mathbf{k}\cdot\mathbf{q})^2/q^2) + D_0'(\mathbf{k}\cdot\mathbf{q})^2/q^2}{(q^2 + m^2)^{d/2+\varepsilon/2}}. \tag{2.7}$$

The integration over $\mathbf{q}$ in (2.7) is performed explicitly:

$$\Sigma_{\theta'\theta}(\omega, k) = -k^2 \, \frac{D_0 \, (d-1) + D_0'}{2d} \, J(m), \tag{2.8a}$$

where we have written

$$J(m) \equiv \int \frac{d\mathbf{q}}{(2\pi)^d} \, \frac{1}{(q^2 + m^2)^{d/2+\varepsilon/2}} = \frac{\Gamma(\varepsilon/2) \, m^{-\varepsilon}}{(4\pi)^{d/2} \Gamma(d/2 + \varepsilon/2)}. \tag{2.8b}$$

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[2] The single-loop approximation to the Dyson equations in the stirred hydrodynamics is equivalent [35] to the well-known direct interaction approximation (DIA) [36]. One can say that in the models (1.1), (1.2), (1.3) or (1.8) the DIA appears to be exact.

Equations (2.5a), (2.8) give an explicit exact expression for the response function in our model; it will be used in Sec. III for the exact calculation of the RG functions. Below we use the intermediate expression (2.7). The integration of Eq. (2.5b) over the frequency ω gives a closed equation for the single-time correlator. Using (2.7) it can be written in the form

$$2\nu_0 k^2 D(k) = C(k) + \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{D_0(k^2 - (\mathbf{k}\cdot\mathbf{q})^2/q^2) + D_0'(\mathbf{k}\cdot\mathbf{q})^2/q^2}{(q^2+m^2)^{d/2+\varepsilon/2}} \left[D(|\mathbf{k}-\mathbf{q}|) - D(k)\right]. \tag{2.9}$$

The function $C(k)$ is supposed to be analytic in k^2, which along with the requirement that $C(k=0)=0$ (so that the Eq. (1.1) has the form of a conservation law for θ) gives

$$C(k) = k^2 \Psi(k) \tag{2.10}$$

with some function $\Psi(k)$, or in the coordinate representation $C(Mr) = -\triangle\Psi(r)$, where $\Psi(r)$ vanishes rapidly for $r\to\infty$.

In the coordinate representation, Eq. (2.9) takes the form

$$2\nu_0 \triangle D(r) = \triangle\Psi + D_0(\delta_{ij}\triangle - \partial_i\partial_j)(A_{ij}D(r)) + D_0'\partial_i\partial_j(A_{ij}D(r)), \tag{2.11}$$

where we have written

$$A_{ij}(\mathbf{r}) \equiv \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{q_i q_j \left[\exp\left(\mathrm{i}\mathbf{q}\cdot\mathbf{r}\right) - 1\right]}{q^2(q^2+m^2)^{d/2+\varepsilon/2}}. \tag{2.12}$$

For $D_0'=0$, Eq. (2.11) coincides (up to the notation) with the well-known equation for the single-time correlator in the model (1.3) obtained in [2].

For $0<\varepsilon<2$, the equations (2.9), (2.12) allow for the limit $m\to 0$: the possible IR divergence of the integrals at $\mathbf{q}=0$ is suppressed by the vanishing of the expressions in the square brackets. In what follows we set $m=0$. Then Eq. (2.12) gives

$$A_{ij}(\mathbf{r}) = -Br^{\varepsilon}\left[\delta_{ij} + \varepsilon r_i r_j/r^2\right], \tag{2.13a}$$

$$B \equiv \frac{-\Gamma(-\varepsilon/2)}{(4\pi)^{d/2}2^{\varepsilon}(d+\varepsilon)\Gamma(d/2+\varepsilon/2)} \tag{2.13b}$$

(note that $B>0$). Using Eq. (2.13) and the fact that the function $D(r)$ depends only on $r=|\mathbf{x}-\mathbf{x}'|$, the differential operators entering into Eq. (2.11) are represented in the form

$$\partial_i\partial_j\left(A_{ij}D(r)\right) = -B(1+\varepsilon)r^{1-d}\partial_r\left(r^{(d-1)/(1+\varepsilon)}\partial_r\left(r^{\varepsilon(d-1)/(1+\varepsilon)}D(r)\right)\right), \tag{2.14a}$$

$$(\delta_{ij}\triangle - \partial_i\partial_j)(A_{ij}D(r)) = (d-1)r^{1-d}\partial_r\left(r^{d-1+\varepsilon}\partial_r D(r)\right), \tag{2.14b}$$

where $\partial_r \equiv \partial/\partial r$; and for the d-dimensional Laplace operator one has:

$$\triangle\Psi(r) = r^{1-d}\partial_r\left(r^{d-1}\partial_r\Psi(r)\right). \tag{2.14c}$$

It then follows from (2.14) that one integration in Eq. (2.11) is readily performed: one can just omit the overall "factor" $r^{1-d}\partial_r r^{d-1}$; the integration constant is determined by the requirement that the solution have no singularity at the origin ($r=0$):

$$2\nu_0\partial_r D = \partial_r\Psi - B(d-1)D_0 r^{\varepsilon}\partial_r D - B(1+\varepsilon)D_0' r^{-\varepsilon(d-1)/(1+\varepsilon)}\partial_r\left(r^{\varepsilon(d+\varepsilon)/(1+\varepsilon)}D\right). \quad (2.15)$$

Equation (2.15) is rewritten in the form

$$\partial_r\left[(1+h_0(\Lambda r)^{\varepsilon})^{\zeta}\ D(r)\right] = [1+h_0(\Lambda r)^{\varepsilon}]^{\zeta-1}\partial_r\tilde{\Psi}, \quad (2.16)$$

where we have denoted

$$h_0 \equiv B\,\frac{(d-1)+\alpha(1+\varepsilon)}{2}, \quad (2.17a)$$

$$\tilde{\Psi} \equiv \Psi/2\nu_0, \quad (2.17b)$$

and the exponent ζ has the form

$$\zeta = \frac{(d+\varepsilon)\,D_0'}{(d-1)D_0+(1+\varepsilon)\,D_0'}, \quad (2.18a)$$

so that

$$\zeta - 1 = \frac{(d-1)\,(D_0'-D_0)}{(d-1)\,D_0+(1+\varepsilon)\,D_0'}. \quad (2.18b)$$

Equation (2.16) is integrated explicitly; the integration constant is found from the requirement that the solution vanish at infinity (including the special case $h_0=0$):

$$D(r) = \frac{-1}{[1+h_0(\Lambda r)^{\varepsilon}]^{\zeta}}\int_r^{\infty} dy\,[1+h_0(\Lambda y)^{\varepsilon}]^{\zeta-1}\partial_y\tilde{\Psi}(y). \quad (2.19)$$

For $D_0'=0$ (so that $\alpha=\zeta=0$), the expression (2.19) reduces (up to the notation) to the well known solution for the purely solenoidal velocity field obtained in [2]. Dimensionality considerations give $\Psi(r)=M^{-2}\psi(Mr)$ with some dimensionless function ψ, see (2.10), so that (2.19) can be rewritten as

$$D(r) = \frac{-1}{2\nu_0 M^2[1+h_0(\Lambda r)^{\varepsilon}]^{\zeta}}\int_{Mr}^{\infty} dy[1+h_0(\Lambda y/M)^{\varepsilon}]^{\zeta-1}\partial_y\psi(y). \quad (2.20)$$

We are interested in the asymptotic form of the correlator $D(r)$ and the structure function $S_2\propto D(0)-D(r)$ in the convective range $\Lambda >> 1/r >> M$, where Λ is determined by (1.4). From (2.20) it then follows

$$D(r=0) \simeq C\,\nu_0^{-1}\,M^{-2-\epsilon(\zeta-1)}\,\Lambda^{\epsilon(\zeta-1)}, \quad (2.21a)$$

where we have used the definitions (1.4) and (2.17), and C is completely dimensionless factor independent of r, M and Λ:

$$C \equiv \frac{-h_0^{\zeta-1}}{2} \int_0^\infty dy y^{\varepsilon(\zeta-1)} \partial_y \psi(y). \tag{2.21b}$$

For the correlator $D(r)$ in the region $\Lambda >> 1/r >> M$ one obtains:

$$D(r) \simeq h_0^{-\zeta} (\Lambda r)^{-\varepsilon\zeta} D(r=0). \tag{2.21c}$$

It follows from (2.21) that $D(r)$ differs from $D(0)$ by the factor $\propto (\Lambda r)^{-\varepsilon\zeta} << 1$. Therefore, the leading contribution to the structure function $S_2 \propto D(0) - D(r)$ in the convective range is given by the constant term $D(0)$, while the r dependent contribution determines only a vanishing correction. Then the comparison of the expression (1.11) for $n = 1$ with the exact result (2.21a) gives $\Delta_2 = -2 + \varepsilon - \varepsilon\zeta$, which along with Eq. (2.18a) leads to the exact expression (1.12) for the critical dimension Δ_2, announced in the Introduction.

The expressions (2.21) simplify for $d = 1$ (and for $D_0 = D_0'$ and any d), when $\zeta = 1$, see (2.18b):

$$D(r) \propto \nu_0^{-1} M^{-2} (\Lambda r)^{-\varepsilon}, \tag{2.22a}$$

$$D(r=0) \propto \nu_0^{-1} M^{-2}. \tag{2.22b}$$

The expression (2.22a) agrees[3] with the result obtained in [20] directly for $d = 1$. In the language of the papers [5–8,20], the leading non-universal term in (2.22a) is related to a nontrivial zero mode of the differential operator entering into Eq. (2.11). We note that the anomalous scaling for the pair correlator with the exactly known exponent was established previously in [30] on the example of a passively advected magnetic field, as a result of the existence of a nontrivial zero mode of the corresponding differential operator. We also note that for the purely solenoidal case, the analogous zero mode is independent of r and cancels out in the structure function, so that the IR behavior of the latter is determined by the universal correction term $r^{2-\varepsilon}$.

In the subsequent Sections, the asymptotic expressions (2.21) will be generalized to the case of higher-order correlators and structure functions.

III. RENORMALIZATION, RG FUNCTIONS, AND RG EQUATIONS

The analysis of the UV divergences in a field theoretical model is based on the analysis of canonical dimensions. Dynamical models of the type (2.1), in contrast to static models, are two-scale, i.e., to each quantity F (a field or a parameter in the action functional) one can assign two independent canonical dimensions, the momentum dimension d_F^k and the frequency dimension d_F^ω, determined from the natural normalization conditions $d_k^k = -d_{\mathbf{x}}^k = 1$, $d_k^\omega = d_{\mathbf{x}}^\omega = 0$, $d_\omega^k = d_t^k = 0$, $d_\omega^\omega = -d_t^\omega = 1$, and from the requirement that each term

[3] In the previous version of this paper, it was erroneously stated that the expression (2.22a) disagrees with the result [20]. The authors are thankful to M. Vergassola for pointing this out to them.

of the action functional be dimensionless (with respect to the momentum and frequency dimensions separately), see e.g. [26,37,38]. Then, based on d_F^k and d_F^ω, one can introduce the total canonical dimension $d_F = d_F^k + 2d_F^\omega$ (in the free theory, $\partial_t \propto \triangle$).

The dimensions for the model (2.1) are given in Table I, including renormalized parameters, which will be considered later on. From Table I it follows that the model is logarithmic (the coupling constant g_0 is dimensionless) at $\varepsilon = 0$, and the UV divergences have the form of the poles in ε in the Green functions. The total dimension d_F plays in the theory of renormalization of dynamical models the same role as does the conventional (momentum) dimension in static problems. The canonical dimensions of an arbitrary 1-irreducible Green function $\Gamma = \langle \Phi \dots \Phi \rangle_{1-\mathrm{ir}}$ are given by the relations

$$d_\Gamma^k = d - N_\Phi d_\Phi, \tag{3.1a}$$

$$d_\Gamma^\omega = 1 - N_\Phi d_\Phi^\omega, \tag{3.1b}$$

$$d_\Gamma = d_\Gamma^k + 2d_\Gamma^\omega = d + 2 - N_\Phi d_\Phi, \tag{3.1c}$$

where $N_\Phi = \{N_\theta, N_{\theta'}, N_{\mathbf{v}}\}$ are the numbers of corresponding fields entering into the function Γ, and the summation over all types of the fields is implied. The total dimension d_Γ is the formal index of the UV divergence. Superficial UV divergences, whose removal requires counterterms, can be present only in those functions Γ for which d_Γ is a non-negative integer.

Analysis of the divergences should be based on the following auxiliary considerations, see [22,26,37,38]:

(i) From the explicit form of the vertex and bare propagators in the model (2.1) it follows that $N_{\theta'} - N_\theta = 2N_0$ for any 1-irreducible Green function, where $N_0 \geq 0$ is the total number of the bare propagators $\langle \theta\theta \rangle_0$ entering into the function (obviously, no diagrams with $N_0 < 0$ can be constructed). Therefore, the difference $N_{\theta'} - N_\theta$ is an even non-negative integer for any nonvanishing function.

(ii) If for some reason a number of external momenta occur as an overall factor in all the diagrams of a given Green function, the real index of divergence d'_Γ is smaller than d_Γ by the corresponding number of unities (the Green function requires counterterms only if d'_Γ is a non-negative integer).

In the model (2.1), the derivative ∂ at the vertex $\theta' \boldsymbol{\partial}(\mathbf{v}\theta)$ can be moved onto the field θ' using the integration by parts, which decreases the real index of divergence: $d'_\Gamma = d_\Gamma - N_{\theta'}$. The field θ' enters into the counterterms only in the form of the derivative $\partial\theta'$.

(iii) A great deal of diagrams in the model (2.1) contain effectively closed circuits of retarded propagators $\langle \theta\theta' \rangle_0$ and therefore vanish. For example, all the nontrivial diagrams of the 1-irreducible function $\langle \theta\theta'\mathbf{v} \rangle_{1-ir}$ vanish.

From the dimensions in Table I we find $d_\Gamma = d + 2 - N_{\mathbf{v}} + N_\theta - (d+1)N_{\theta'}$ and $d'_\Gamma = (d+2)\,(1 - N_{\theta'}) - N_{\mathbf{v}} + N_\theta$. From these expressions it follows that for any d, superficial divergences can only exist in the 1-irreducible functions with $N_{\theta'} = 1$, $N_{\mathbf{v}} = N_\theta = 0$ ($d_\Gamma = 1$, $d'_\Gamma = 0$), $N_{\theta'} = N_{\mathbf{v}} = N_\theta = 1$ ($d_\Gamma = 1$, $d'_\Gamma = 0$), and $N_{\theta'} = N_\theta = 1$, $N_{\mathbf{v}} = 0$ ($d_\Gamma = 2$, $d'_\Gamma = 1$) (we recall that $N_\theta \leq N_{\theta'}$, see (i) above). However, no diagrams can be constructed for the first of these functions, while for the second function, all the nontrivial diagrams vanish (see (iii) above). As in the case of the purely solenoidal field [22], we are left with

the only superficially divergent function $\langle\theta'\theta\rangle_{1-ir}$; the corresponding counterterm necessarily contains the factor of $\partial\theta'$ and is therefore reduced to $\theta'\triangle\theta$. Introduction of this counterterm is reproduced by the multiplicative renormalization of the parameters g_0, ν_0 in the action functional (2.1) with the only independent renormalization constant Z_ν:

$$\nu_0 = \nu Z_\nu, \tag{3.2a}$$

$$g_0 = g\mu^\varepsilon Z_g, \tag{3.2b}$$

$$Z_g = Z_\nu^{-1}. \tag{3.2c}$$

Here μ is the renormalization mass in the minimal subtraction scheme (MS), which we always use in what follows, g and ν are renormalized analogues of the bare parameters g_0 and ν_0, and $Z = Z(g, \alpha, \varepsilon, d)$ are the renormalization constants. Their relation in (3.2c) results from the absence of renormalization of the contribution with D_v in (2.1), so that $D_0 \equiv g_0\nu_0 = g\mu^\varepsilon\nu$. No renormalization of the fields and the parameters m, M, α is required, i.e., $Z_\Phi = 1$ for all Φ and $m_0 = m$, $Z_m = 1$, etc. The renormalized action functional has the form

$$S_{ren}(\Phi) = \theta' D_\theta \theta'/2 + \theta' \left[-\partial_t\theta - \boldsymbol{\partial}(\mathbf{v}\theta) + \nu Z_\nu \triangle\theta\right] - \mathbf{v} D_v^{-1}\mathbf{v}/2, \tag{3.3}$$

where the contribution with D_v is expressed in renormalized parameters using (3.2).

The relation $S(\Phi, e_0) = S_{ren}(\Phi, e, \mu)$ (where e_0 is the complete set of bare parameters, and e is the set of renormalized parameters) for the generating functional $W(A)$ in (2.2) yields $W(A, e_0) = W_{ren}(A, e, \mu)$. We use $\widetilde{\mathcal{D}}_\mu$ to denote the differential operation $\mu\partial_\mu$ for fixed e_0 and operate on both sides of this equation with it. This gives the basic RG differential equation:

$$\mathcal{D}_{RG} W_{ren}(A, e, \mu) = 0, \tag{3.4a}$$

where $\mathcal{D}_{RG}$ is the operation $\widetilde{\mathcal{D}}_\mu$ expressed in the renormalized variables:

$$\mathcal{D}_{RG} \equiv \mathcal{D}_\mu + \beta(g)\partial_g - \gamma_\nu(g)\mathcal{D}_\nu, \tag{3.4b}$$

where we have written $\mathcal{D}_x \equiv x\partial_x$ for any variable x, and the RG functions (the β function and the anomalous dimension γ) are defined as

$$\gamma_\nu(g) \equiv \widetilde{\mathcal{D}}_\mu \ln Z_\nu, \tag{3.5a}$$

$$\beta(g) \equiv \widetilde{\mathcal{D}}_\mu g = g\,(-\varepsilon + \gamma_\nu). \tag{3.5b}$$

The relation between β and γ in (3.5b) results from the definitions and the relation (3.2c).

The renormalization constant Z_ν is found from the requirement that the 1-irreducible function $\langle\theta'\theta\rangle_{1-ir}$ expressed in renormalized variables be UV finite (i.e., be finite for $\varepsilon \to 0$). This requirement determines Z_ν up to an UV finite contribution; the latter is fixed by the choice of a renormalization scheme. In the MS scheme all renormalization constants have

the form "1 + only poles in ε." The function $G^{-1} = \langle\theta'\theta\rangle_{1-ir}$ in our model is known exactly, see Eqs. (2.5a), (2.8). Let us substitute (3.2) into Eqs. (2.5a), (2.8) and choose Z_ν to cancel the pole in ε in the integral $J(m)$. This gives:

$$Z_\nu = 1 - g\, C_d \frac{d-1+\alpha}{2d\varepsilon}, \tag{3.6}$$

where we have written $C_d \equiv S_d/(2\pi)^d$ and $S_d \equiv 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in d-dimensional space. Note that the result (3.6) is exact, i.e., it has no corrections of order g^2, g^3, and so on; this is a consequence of the fact that the single-loop approximation (2.8) for the response function is exact. Note also that for $\alpha = 0$ (3.6) coincides with the exact expression for Z_ν in the "incompressible" case obtained in [22].

For the anomalous dimension $\gamma_\nu(g) \equiv \widetilde{\mathcal{D}}_\mu \ln Z_\nu = \beta(g)\partial_g \ln Z_\nu$ from the relations (3.5b) and (3.6) one obtains:

$$\gamma_\nu(g) = \frac{-\varepsilon\mathcal{D}_g \ln Z_\nu}{1-\mathcal{D}_g \ln Z_\nu} = g\, C_d \frac{d-1+\alpha}{2d}. \tag{3.7}$$

From (3.5b) it then follows that the RG equations of the model have an IR stable fixed point $[\beta(g_*) = 0,\ \beta'(g_*) > 0]$ with the coordinate

$$g_* = \frac{2d\varepsilon}{C_d(d-1+\alpha)}. \tag{3.8}$$

The fixed point is degenerated: its coordinate g_* depends continuously on the parameter $\alpha = D'_0/D_0$.[4] The value of $\gamma_\nu(g)$ at the fixed point is also found exactly:

$$\gamma^*_\nu \equiv \gamma_\nu(g_*) = \varepsilon. \tag{3.9}$$

The solution of the RG equations on the example of the stochastic hydrodynamics is discussed in detail in Refs. [26,38]; see also [22] for the case of the model (1.3); below we confine ourselves to the only information we need.

In general, if some quantity F (a parameter, a field or composite operator) is renormalized multiplicatively, $F = Z_F F_{ren}$ with certain renormalization constant Z_F, its critical dimension is given by the expression (cf. [26,37,38]):

$$\Delta[F] \equiv \Delta_F = d^k_F + \Delta_\omega d^\omega_F + \gamma^*_F, \tag{3.10}$$

where d^k_F and d^ω_F are the corresponding canonical dimensions, γ^*_F is the value of the anomalous dimension $\gamma_F(g) \equiv \widetilde{\mathcal{D}}_\mu \ln Z_F$ at the fixed point, and $\Delta_\omega = -2 + \gamma^*_\nu = -2 + \varepsilon$ is the critical dimension of frequency. The critical dimensions of the fields Φ in our model are found exactly; they are independent of the parameter α and coincide with their analogues in the model (1.3), cf. [22]:

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[4] Formally, α can be treated as the second coupling constant. The corresponding beta-function $\beta_\alpha \equiv \widetilde{\mathcal{D}}_\mu \alpha$ vanishes identically owing to the fact that α is not renormalized. Therefore, the equation $\beta_\alpha = 0$ gives no additional constraint on the values of the parameters g, α at the fixed point.

$$\Delta_{\mathbf{v}} = 1 - \varepsilon, \tag{3.11a}$$

$$\Delta_\theta = -1 + \varepsilon/2, \tag{3.11b}$$

$$\Delta_{\theta'} = d + 1 - \varepsilon/2 \tag{3.11c}$$

(we recall that the fields in the model (2.1) are not renormalized and therefore $\gamma_\Phi = 0$ for all Φ).

Let $G(r) = \langle F_1(x) F_2(x') \rangle$ be a single-time two-point quantity, for example, the pair correlation function of the primary fields $\Phi \equiv \{\theta, \theta', \mathbf{v}\}$ or some multiplicatively renormalizable composite operators. The existence of the IR stable fixed point implies that in the IR asymptotic region $\Lambda r >> 1$ and any fixed Mr the function $G(r)$ is found in the form

$$G(r) \simeq \nu_0^{d_G^\omega} \, \Lambda^{d_G} (\Lambda r)^{-\Delta_G} \, \xi(Mr), \tag{3.12}$$

with certain, as yet unknown, scaling function ξ of the critically dimensionless argument Mr. The canonical dimensions d_G^ω, d_G and the critical dimension Δ_G ot the function $G(r)$ are equal to the sums of the corresponding dimensions of the quantities F_i.

Now let us turn to the composite operators of the form $\theta^n(x)$ entering into the structure functions (1.5) and the correlators (1.9).

In general, counterterms to a given operator F are determined by all possible 1-irreducible Green functions with one operator F and arbitrary number of primary fields, $\Gamma = \langle F(x) \Phi(x_1) \dots \Phi(x_2) \rangle_{1-ir}$. The total canonical dimension (formal index of divergence) for such functions is given by

$$d_\Gamma = d_F - N_\Phi d_\Phi, \tag{3.13}$$

with the summation over all types of fields entering into the function. For superficially divergent diagrams, the real index $d'_\Gamma = d_\Gamma - N_{\theta'}$ is a non-negative integer. From Table I and Eq. (3.13) for the operators $\theta^n(x)$ we obtain $d_F = -n$, $d_\Gamma = -n + N_\theta - N_{\mathbf{v}} - (d+1) N_{\theta'}$, and $d'_\Gamma = -n + N_\theta - N_{\mathbf{v}} - (d+2) N_{\theta'}$. From the analysis of the diagrams it follows that the total number of the fields θ entering into the function Γ can never exceed the number of the fields θ in the operator θ^n itself, i.e., $N_\theta \leq n$. Therefore, the divergence can only exist in the functions with $N_{\mathbf{v}} = 0$, $N_{\theta'} = 0$, and arbitrary value of $n = N_\theta$, for which $d_\Gamma = d'_\Gamma = 0$ and the corresponding counterterm has the form θ^n. It then follows that the operator θ^n is renormalized multiplicatively, $\theta^n = Z_n [\theta^n]_{ren}$.

Note an important difference between the case of a purely transversal velocity field (1.3) and the general case (1.8). In the first case, the derivative $\boldsymbol{\partial}$ at the vertex can be moved onto the field θ owing to the transversality of the velocity field, $\theta' \boldsymbol{\partial}(\mathbf{v}\theta) = \theta'(\mathbf{v}\boldsymbol{\partial})\theta$. This reduces the real index d'_Γ by at least one unity, so that d'_Γ becomes strictly negative, see [22]. This means that the operator θ^n requires no counterterms at all, i.e., it is in fact UV finite, $Z_n = 1$.[5] It then follows that the critical dimension of $\theta^n(x)$ in the model (1.3) is simply

[5] This "non-renormalization" result can be interpreted as the fact that the scalar field remains a

given by the expression (3.10) with no correction from γ_F^* and is therefore reduced to the sum of the critical dimensions of the factors [22]:

$$\Delta_n \equiv \Delta[\theta^n] = n\Delta[\theta] = n(-1+\varepsilon/2). \tag{3.14}$$

In the general case (1.8), the constants Z_n are nontrivial, and the simple relation (3.14) is no longer valid. The two-loop calculation of the constants Z_n is explained in detail in Sec. V, and here we only give the two-loop result for the critical dimensions Δ_n in the model (2.1):

$$\Delta_n = n(-1+\varepsilon/2) - \frac{\alpha n(n-1)d\varepsilon}{2(d-1+\alpha)} + \frac{\alpha(\alpha-1)n(n-1)(d-1)\varepsilon^2}{2(d-1+\alpha)^2} + \\ + \frac{\alpha^2 n(n-1)(n-2)d\, h(d)\varepsilon^2}{4(d-1+\alpha)^2} + O(\varepsilon^3), \tag{3.15}$$

where we have denoted

$$h(d) \equiv \sum_{k=0}^{\infty} \frac{k!}{4^k(d/2+1)\dots(d/2+k)} = F(1,1;d/2+1;1/4), \tag{3.16}$$

and $F(\dots)$ is the hypergeometric series, see [39].

In the special case $n=2$ one obtains from (3.15):

$$\Delta_2 = -2 - \frac{\varepsilon(d-1)(\alpha-1)}{(d-1+\alpha)} + \frac{\varepsilon^2(d-1)\alpha(\alpha-1)}{(d-1+\alpha)^2} + O(\varepsilon^3). \tag{3.17}$$

Expression (3.15) is simplified for any integer value of d owing to the fact that the series in (3.16) reduces then to a finite sum, see [39]:

$$h(d) = 2d\left[(-3)^{d/2-1}\ln(4/3) + \sum_{k=2}^{d/2}\frac{(-3)^{k-2}}{d/2-k+1}\right] \tag{3.18a}$$

for any even value of d and

$$h(d) = 2d\left[(-1)^{(d-1)/2}\cdot 3^{d/2-2}\cdot\pi + 2\sum_{k=1}^{(d-1)/2}\frac{(-3)^{(d-1)/2-k}}{2k-1}\right] \tag{3.18b}$$

for any odd value of d, which gives $h(d) = 2\pi/(3\sqrt{3})$ for $d=1$, $h(d) = 4\ln(4/3)$ for $d=2$, and $h(d) = 12 - 2\pi\sqrt{3}$ for $d=3$. (We note that for $d=1$ and $d=2$ the sums in (3.18) contain no terms). The case of a purely potential velocity field is obtained for $D_0' = \text{const}$,

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continuous function even in the limit $\nu_0 \to 0$ or equivalently $\Lambda \to \infty$. The nontrivial UV renormalization of the monomials $(\partial\theta\partial\theta)^n$ [22] points to the fact that the scalar field is not differentiable, i.e., its gradients exist only as distributions. One of the authors (N.V.A) is thankful to G.L.Eyink for pointing this out to him, see also Refs. [12–14].

$D_0 = 0$ or, equivalently, $\alpha \to \infty$, $g_0' \equiv g_0\alpha = \mathrm{const}$. From (3.8) it then follows that at the fixed point $g_*' = 2d\varepsilon/C_d$; the values of the critical dimensions Δ_n are obtained simply by taking the limit $\alpha \to \infty$ in the expressions (3.15), (3.17) and have the form

$$\Delta_n = n(-1+\varepsilon/2) - n(n-1)d\varepsilon/2 + n(n-1)(d-1)\varepsilon^2/2 + \\ + n(n-1)(n-2)\,h(d)\,d\varepsilon^2/4 + O(\varepsilon^3). \tag{3.19}$$

In the special case $d = 1$ one obtains

$$\Delta_n = -n + n\varepsilon - n^2\varepsilon/2 + n(n-1)(n-2)\varepsilon^2\pi/(6\sqrt{3}) + O(\varepsilon^3). \tag{3.20}$$

For the pair correlators of the operators θ^n we obtain from Table I and Eqs. (3.12), (3.15):

$$\langle\theta^n(x)\theta^p(x')\rangle = \nu_0^{-(n+p)/2}\,\Lambda^{-(n+p)}(\Lambda r)^{-\Delta_n-\Delta_p}\xi_{n,p}(Mr), \tag{3.21}$$

with the dimensions Δ_n given in (3.15) and certain scaling functions $\xi_{n,p}(Mr)$ (for odd $n+p$ they vanish). We recall that the representation (3.21) holds for $\Lambda r >> 1$ and any fixed Mr; the behavior of the functions $\xi_{n,p}(Mr)$ for $Mr << 1$ (convective range) is studied in the subsequent section.

IV. OPERATOR PRODUCT EXPANSION AND ANOMALOUS SCALING

The representation (3.21) for any functions $\xi_{n,p}(Mr)$ correspond to IR scaling in the region $\Lambda r >> 1$ and any fixed Mr with definite critical dimensions Δ_n given in (3.15). The expressions (1.9) should be understood as certain additional statements about the explicit form of the asymptotic behavior of the functions $\xi_{n,p}(Mr)$ for $Mr \to 0$. The form of the scaling functions $\xi_{n,p}(Mr)$ in the representation (3.21) is not determined by the RG equations themselves; these functions can be calculated in the form of series in ε. However, this ε expansion is not suitable for the analysis of their behavior for $Mr \to 0$, because the actual expansion parameter appears to be $\varepsilon \ln(Mr)$ rather than ε itself, cf. [22,26,38]. In contrast to the "large UV logarithms" $\ln(\Lambda r)$, the summation of these "large IR logarithms" is not performed automatically by the solution of the RG equations.

In the theory of critical phenomena, the asymptotic form of scaling functions for $M \to 0$ is studied using the well known Wilson operator product expansion (OPE), see e.g. [25]; the analogue of $L \equiv M^{-1}$ is there the correlation length r_c. This technique is also applied to the theory of turbulence, see e.g. [22,26,38].

According to the OPE, the single-time product $F_1(x)F_2(x')$ of two renormalized operators at $\mathbf{x} \equiv (\mathbf{x}+\mathbf{x}')/2 = \mathrm{const}$, and $\mathbf{r} \equiv \mathbf{x}-\mathbf{x}' \to 0$ has the representation

$$F_1(x)F_2(x') = \sum_\alpha C_\alpha(\mathbf{r})F_\alpha(\mathbf{x},\mathbf{t}), \tag{4.1}$$

in which the functions C_α are the Wilson coefficients regular in M^2 and F_α are all possible renormalized local composite operators allowed by symmetry, with definite critical dimensions Δ_α.

The renormalized correlator $\langle F_1(x)F_2(x')\rangle$ is obtained by averaging (4.1) with the weight $\exp S_R$, the quantities $\langle F_\alpha\rangle$ appear on the right hand side. Their asymptotic behavior for $M\to 0$ is found from the corresponding RG equations and has the form

$$\langle F_\alpha\rangle \propto M^{\Delta_\alpha}. \tag{4.2}$$

From the operator product expansion (4.1) we therefore find the following expression for the scaling function $\xi(Mr)$ in the representation (3.12) for the correlator $\langle F_1(x)F_2(x')\rangle$:

$$\xi(Mr) = \sum_\alpha A_\alpha\,(Mr)^{\Delta_\alpha}, \tag{4.3}$$

with coefficients $A_\alpha = A_\alpha(Mr)$, which are regular in $(Mr)^2$, generated by the Wilson coefficients C_α in (4.1).

We note that for a Galilean invariant product $F_1(x)F_2(x')$, the right hand side of Eq. (4.1) can involve any Galilean invariant operator, including tensor operators, whose indices are contracted with the analogous indices of the coefficients C_α. Without loss of generality, it can be assumed that the expansion is made in irreducible tensors, so that only scalars contribute to the correlator $\langle F_1F_2\rangle$ because the averages $\langle F_\alpha\rangle$ for non-scalar irreducible tensors vanish. For the same reason, the contributions to the correlator from all operators of the form ∂F with external derivatives vanish owing to translational invariance.

The leading contributions for $Mr\to 0$ are those with the smallest dimension Δ_α, and in the ε expansions they are those with the smallest $d_\alpha \equiv d[F_\alpha]$ for $\varepsilon=0$. In the standard model ϕ^4 of the theory of critical behavior one has $\Delta_\alpha = n_\alpha + O(\varepsilon)$, where $n_\alpha \geq 0$ is the total number of fields and derivatives in F_α. The operator $F=1$ has the smallest value $n_\alpha = 0$, and it gives a contribution to (4.3) which is regular in $(Mr)^2$ and has a finite limit as $Mr\to 0$. The first nontrivial contribution is generated by the operator ϕ^2 with $n_\alpha = 2$. It has the form $(Mr)^{2+O(\varepsilon)}$ and only determines a correction, vanishing at $Mr\to 0$, to the leading term generated by the operator $F=1$.

The distinguishing feature of the models describing turbulence is the existence of "dangerous" composite operators with *negative* critical dimensions [22,26,38]. The contributions of the dangerous operators into the operator product expansions lead to a singular behavior of the scaling functions on M for $Mr\to 0$. It is obvious from (3.15) that all the operators θ^n in the model (2.1) are dangerous at least for small ε, and the spectrum of their critical dimensions is unbounded from below. If all these operators contributed to the OPE like (4.1), the analysis of the small M behavior would imply the summation of their contributions. Such a summation is indeed required for the case of the different-time correlators in the stochastic Navier–Stokes equation, and it establishes the substantial dependence of the correlators on M and their superexponential decay as the time differences increase, see [26,38]. Fortunately, the problem simplifies for the model (2.1).

From the analysis of the diagrams it follows that the number of the fields θ in the operator F_α entering into the right hand sides of the expansions (4.1) can never exceed the total number of the fields θ in their left hand sides. Therefore, only finite number of operators θ^n contribute to each operator product expansion, and the asymptotic form of the scaling functions is simply determined by the operator θ^n with the lowest critical dimension, i.e., with the largest possible number of the fields θ. For the scaling functions $\xi_{n,p}(Mr)$ entering into the expressions (3.21) this gives $\xi_{n,p}(Mr) \propto (Mr)^{\Delta_{n+p}}$, which lead to the asymptotic expression (1.9) announced in the Introduction.

It is noteworthy that the set of the operators θ^n is "closed with respect to the fusion" in the sense that the leading term in the OPE for the pair correlator $\langle \theta^n \theta^p \rangle$ is given by the operator θ^{n+p} from the same family with the summed index $n+m$. This fact along with the inequality

$$\Delta_n + \Delta_p > \Delta_{n+p}, \tag{4.4}$$

which is obvious from the explicit expression (3.15) for small values of ε, can be interpreted as the statement that the correlations of the scalar field θ in the model (2.1) exhibit multifractal behavior, see [40–42]. In the case of the solenoidal velocity field, the dimension Δ_n becomes linear in n, see (3.14), and the relation (1.9) reduces to the so-called "gap scaling" (see [40]). In this case, the nontrivial multifractal behavior manifests itself in the correlations of the dissipation rate rather than in the correlations of the field itself, see [22].

Now let us turn to the structure functions (1.5) in the convective range $\Lambda r >> 1$, $Mr << 1$. From the expression (1.9) it follows

$$S_{2n} \simeq \nu_0^{-n}\, \Lambda^{-2n}\, (M/\Lambda)^{\Delta_{2n}} \left[1 + \sum_{\substack{k+p=2n \\ k,p\neq 0}} c_{kp} (\Lambda r)^{\Delta_{2n}-\Delta_k-\Delta_p} \right], \tag{4.5}$$

where the coefficients c_{kp} are independent of the scales Λ, M and the separation r. It is obvious from the inequality (4.4) that all the contributions in the sum in (4.5) vanish in the region $\Lambda r >> 1$, so that the leading terms of the structure functions do not depend on r and are given by the Eq. (1.11).

The comparison of the expressions (1.9) and (1.11) for $k = p = 1$ with the exact results (2.21c) and (2.21a) gives $\Delta_2 = -2 + \varepsilon - \varepsilon\zeta$, which along with Eq. (2.18a) leads to the exact expression (1.12) for the critical dimension Δ_2, announced in the Introduction. We note that the expression (3.17) for Δ_2 obtained within the RG approach is in agreement with the corresponding terms of the expansion in ε of the exact exponent (1.12) for all d and α. We also note that (2.21c) is consistent with the exact RG result $\Delta_\theta = -1 + \varepsilon/2$, see (3.11b).

It is seen from (4.5) that the IR behavior for the structure functions is determined by the contributions of the composite operators θ^n to the corresponding OPE. The operators θ^n obviously do not appear in the naïve Taylor expansions of the structure functions (1.5) for $r \to 0$: the Taylor expansion for the function S_{2n} starts with the monomial $(\partial_i\theta\partial_i\theta)^n$. However, the operators entering into operator product expansions are not only those which appear in the Taylor expansions, but also all possible operators which admix to them in renormalization. One can easily check that all the monomials θ^{2p} with $p \leq n$ admix to $(\partial_i\theta\partial_i\theta)^n$ in renormalization. As a result, their contributions appear in the OPE for the structure functions and dominate their IR asymptotic behavior.

The situation changes if the velocity field is purely solenoidal, with the correlator given in (1.3). In this case, the field θ enters into the vertex in the form of a derivative, $\theta' \boldsymbol{\partial}(\mathbf{v}\theta) = \theta'(\mathbf{v}\boldsymbol{\partial})\theta$, and therefore only derivatives of θ can appear in the counterterms to the monomials $(\partial_i\theta\partial_i\theta)^n$. Hence, the operators of the form θ^n cannot admix in renormalization to the monomials $(\partial_i\theta\partial_i\theta)^n$ and cannot appear in the OPE for the structure functions (1.5). This means that the contributions of the operators θ^n to the pair correlators (1.9) cancel out in the structure functions, and the IR behavior of the latter is dominated by the operators

$(\partial_i\theta\partial_i\theta)^n$; see (1.6). The cancellation becomes possible due to the fact that the dimension Δ_n for $\alpha = 0$ is a function linear in n, see (3.14), and therefore all the terms in the square brackets in (4.5) are independent of Λr. In this case, the anomalous exponents are determined by the critical dimensions of the powers of the operator $\partial_i\theta\partial_i\theta$; these dimensions are known up to the order ε^2 of the ε expansion [22].

For $d = 1$, the behavior analogous to (1.6) in the model (1.8) is demonstrated by the structure functions of the field $\phi(t, x)$ defined so that $\theta(t, x) = \partial_x\phi(t, x)$. In this formulation, the problem was studied in the paper [20] within the zero-modes approach and using numerical simulations. The structure functions of the "potential" ϕ are not simply related to the structure functions of the primary field θ, but they can be derived directly using the RG technique. Obviously, the field ϕ enters into the vertex in the form of the derivative $\theta'\partial_x(v\partial_x\phi)$. Therefore, the operators ϕ^n are not renormalized and their critical dimensions are given by the relations analogous to (3.14): $\Delta[\phi^n] = n\Delta[\phi]$, where $\Delta[\phi] = -1 + \Delta[\theta] = -2 + \varepsilon/2$, see Eq. (3.11b). The structure functions are then given by the expression analogous to (1.8):

$$S_{2n} \simeq D_0^{-n} r^{n(4-\varepsilon)} (Mr)^{\Delta_{2n}},$$

where the part of the anomalous exponents is played by the critical dimensions Δ_{2n} of the operators $(\partial_x\phi\partial_x\phi)^n \equiv \theta^{2n}$ given by Eq. (3.20). In the notation of [20] we then have $\zeta_{2n} = n(4 - \varepsilon) + \Delta_{2n} = 2n - n\varepsilon(2n - 1) - 2\pi n(n - 1)(2n - 1)\varepsilon^2/3\sqrt{3}$, in agreement with the $O(\varepsilon)$ result obtained in [20] using the zero-modes approach; the exponent $\zeta_2 = 2 - \varepsilon$ is exact.

Let us conclude this section with a brief discussion of the simple model of a passively advected magnetic field considered in [30].[6] In this case, both $\theta \equiv \theta_i(x)$ and the velocity are solenoidal vector fields. The velocity field is taken to be Gaussian with the correlator (1.3), and the nonlinearity in (1.1) has the form $v_j\partial_j\theta_i - \theta_j\partial_j v_i$. The anomalous scaling in this model also appears already for the pair correlator; the corresponding exponent is found exactly [30].

The RG analysis given above and in [22] is extended directly to this model. It turns out that the expressions for the renormalization constant Z_ν, the RG functions β and γ_ν, the fixed point g_*, and the critical dimension Δ_θ coincide exactly with the expressions (3.5)–(3.8) and (3.11b) for the model (2.1) with the substitution $\alpha = 0$. For the IR asymptotic region, the expressions of the form (3.12) are obtained for the correlation functions of various composite operators; the corresponding critical dimensions Δ_F are calculated in the form of the ε expansions. In particular, for the critical dimensions Δ_{2n} of the scalar operators $\theta^{2n} \equiv (\theta_i\theta_i)^n$ we have obtained

$$\Delta_{2n} = -2n - \frac{2n(n-1)\varepsilon}{d+2} + O(\varepsilon^2), \tag{4.6}$$

and for the special case $n = 1$ we have

$$\Delta_2 = -2 - \frac{2(d-2)\varepsilon^2}{d(d-1)} + O(\varepsilon^3). \tag{4.7}$$

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[6] In more realistic models of the MHD turbulence the magnetic field indeed behaves as a passive vector in the so-called kinetic fixed point of the RG equations, see [43,44].

For the dimensions Δ'_{2n} of the second-rank irreducible tensors $\theta_i\theta_j\theta^{2n-2} - \delta_{ij}\theta^{2n}/d$ we have

$$\Delta'_{2n} = -2n + \frac{\varepsilon[d(d+1) - 2(d-1)n(n-1)]}{(d-1)(d+2)} + O(\varepsilon^2). \tag{4.8}$$

The leading terms of the small Mr behavior of the scaling functions are determined by the contributions of the scalar operators θ^{2n}, and the part of the anomalous exponents is played by the dimensions (4.6). For the special case of the pair correlator it then follows $\langle\theta(x)\theta(x')\rangle \propto (\Lambda r)^{-2\Delta_\theta}(Mr)^{\Delta_2}$. In the notation of [30] we have $\gamma = \Delta_2 - 2\Delta_\theta$; from Eqs. (3.11b) and (4.7) it follows $\Delta_2 - 2\Delta_\theta = -\varepsilon - 2\varepsilon^2(d-2)/d(d-1) + O(\varepsilon^3)$ for any d, in agreement with the exact expression for γ obtained in [30].

V. CALCULATION OF THE ANOMALOUS EXPONENTS TO THE ORDER ε^2

In this section we present the two-loop calculation of the critical dimensions Δ_n of the composite operators θ^n, which determine the anomalous exponents in the expressions (1.9), (1.11).

The operators θ^n are renormalized multiplicatively, $\theta^n = Z_n[\theta^n]_{ren}$ (see Sec. III). The renormalization constants Z_n can be found from the requirement that the 1-irreducible correlation function

$$\langle[\theta^n]_{ren}(x)\theta(x_1)\dots\theta(x_2)\rangle_{1-ir} = Z_n^{-1}\langle\theta^n(x)\theta(x_1)\dots\theta(x_2)\rangle_{1-ir} \equiv Z_n^{-1}\Gamma_n \tag{5.1}$$

be UV finite, i.e., have no poles in ε, when expressed in renormalized variables using the formulas (3.2). This requirement determines Z_n up to an UV finite part; the choice of the finite part depends on the “subtraction scheme”. Most convenient for practical calculations is the minimal subtraction (MS) scheme. In the MS scheme, only poles in ε are subtracted from the divergent expressions, and the renormalization constants have the form “1 + only poles in ε”. In particular,

$$Z_n^{-1} = 1 + \sum_{k=1}^{\infty} a_k(g)\varepsilon^{-k} = 1 + \sum_{n=1}^{\infty} g^n \sum_{k=1}^{n} a_{nk}\varepsilon^{-k}. \tag{5.2}$$

The coefficients a_{nk} in our model depend only on the space dimension d and the completely dimensionless parameter α; their independence of ε is a feature specific for the MS scheme. One-loop diagrams generate contributions of order g in (5.2), two-loop ones generate contributions of order g^2, and so on. The order of the pole in ε does not exceed the number of the loops in the diagram.

The two-loop diagrams of the function Γ_n required for the calculation of Z_n to the order g^2 and the corresponding symmetry coefficients are given in Table II. The solid lines in the diagrams denote the bare propagator $\langle\theta\theta'\rangle_0$ from (2.4a), the end with a slash correspond to the field θ', and the end without a slash correspond to θ; the dashed lines denote the bare propagator (1.8). Note that the propagator $\langle\theta\theta\rangle_0$ does not enter into the diagrams for Γ_n. The black circle with $p \geq 0$ attached “legs” denotes the vertex factor F_p given by the p-fold variational derivative $F_p \equiv \delta\theta^n(x)/\delta\theta(x_1)\dots\delta\theta(x_p)$.

Now let us turn to the calculation of the diagrams from Table II. It is sufficient to calculate the function Γ_n in the momentum-frequency representation with all the external

momenta and frequencies equal to zero; the IR regularization is then provided by the "mass" m from the correlator (1.8). In what follows, we use the notation

$$R_{ij}(\mathbf{k}) \equiv D_0 P_{ij}(\mathbf{k}) + D_0' Q_{ij}(\mathbf{k}) \tag{5.3a}$$

and

$$S(k) \equiv (k^2+m^2)^{-d/2-\varepsilon/2}. \tag{5.3b}$$

We also recall the relations $D_0 = g_0\nu_0$ and $\alpha = D_0'/D_0$.

The diagram D_2 differs from D_1 only by the insertion of the simplest self-energy diagram $\Sigma_{\theta\theta'}$ into one of the two lines $\langle\theta\theta'\rangle$. Therefore, the combination D_1+2D_2 entering into Γ_n can be easily calculated as a whole: we calculate the single-loop diagram D_1 with the *exact* propagators $\langle\theta\theta'\rangle$ instead of the bare propagators $\langle\theta\theta'\rangle_0$ and then expand the result in g_0 to the order g_0^2. From the exact solution (see Sec. II) it follows that the propagator $\langle\theta\theta'\rangle$ is obtained from its bare counterpart simply by the replacement $\nu_0 \to \eta_0$, where the exact "effective diffusivity" has the form[7]

$$\eta_0 \equiv \nu_0 + \frac{D_0(d-1)+D_0'}{2d} J(m),$$

see (2.5a), (2.8). Then the "exact" analogue of the diagram D_1 is given by

$$\int \frac{d\omega}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{S(k)\, R_{ij}\,(\mathbf{k}) k_i k_j}{|\mathrm{i}\omega + \eta_0 k^2|^2} = D_0' J(m)/2\eta_0, \tag{5.4}$$

where we have performed the elementary integration over the frequency and used the isotropy of the function $S(k)$. The expansion of the result (5.4) in g_0 gives

$$D_1 + 2D_2 = \frac{\alpha g_0 J(m)}{2}\left[1 - \frac{g_0(d-1+\alpha)J(m)}{2d}\right]. \tag{5.5}$$

The right hand side of Eq. (5.5) is expressed in renormalized variables by the substitution $g_0 = g\mu^\varepsilon Z_\nu^{-1}$ with the constant Z_ν from (3.6), which within our accuracy gives:

$$\begin{aligned} D_1 + 2D_2 = \frac{\alpha g \mu^\varepsilon J(m)}{2} + \frac{\alpha\, g^2(d-1+\alpha)\mu^\varepsilon J(m)}{4d}\,[C_d/\varepsilon - \mu^\varepsilon J(m)] \equiv \\ \equiv g\, D^{(1)} + g^2\, D^{(2)}. \end{aligned} \tag{5.6}$$

The diagram D_3 is represented by the integral

$$D_3 = \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^d} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{R_{ij}(\mathbf{q})(k+q)_i(k+q)_j R_{ps}(\mathbf{k}) k_p k_s}{|\mathrm{i}\omega + \nu_0(\mathbf{k}+\mathbf{q})^2|^2\,|\mathrm{i}\omega' + \nu_0 k^2|^2} S(k) S(q), \tag{5.7}$$

and the integrations over the frequencies give

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[7] See [45] for the exact expression for the effective diffusivity in the incompressible case.

$$D_3 = \frac{\alpha g_0^2}{4} \int \frac{d\mathbf{k}}{(2\pi)^d} \int \frac{d\mathbf{q}}{(2\pi)^d} \left[\alpha + (1-\alpha) \frac{k^2 \sin^2\vartheta}{(\mathbf{k}+\mathbf{q})^2} \right] S(k)S(q), \tag{5.8}$$

where ϑ is the angle between the vectors $\mathbf{k}$ and $\mathbf{q}$, so that $\mathbf{k}\cdot\mathbf{q} = kq\cos\vartheta$. The symmetry of the integral (5.8) in $\mathbf{k}$ and $\mathbf{q}$ allows one to perform the substitution $k^2 \to (\mathbf{k}+\mathbf{q})^2/2 - \mathbf{k}\cdot\mathbf{q}$ in the integrand, which gives:

$$D_3 = \frac{\alpha^2 g_0^2}{4} J^2(m) + \frac{\alpha(1-\alpha) g_0^2}{8} [J_1(m) - 2J_2(m)], \tag{5.9}$$

where we have written

$$J_1(m) \equiv \int \frac{d\mathbf{k}}{(2\pi)^d} \int \frac{d\mathbf{q}}{(2\pi)^d} \sin^2\vartheta\, S(k)S(q) \tag{5.10}$$

and

$$J_2(m) \equiv \int \frac{d\mathbf{k}}{(2\pi)^d} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{\mathbf{k}\cdot\mathbf{q}\sin^2\vartheta}{(\mathbf{k}+\mathbf{q})^2} S(k)S(q). \tag{5.11}$$

The integral in (5.10) can be easily expressed via $J(m)$:

$$\begin{aligned} J_1(m) = C_d^2 \int_0^\infty dk\, k^{d-1} \int_0^\infty dq\, q^{d-1} \int d\mathbf{n} \sin^2\vartheta\, S(k)S(q) = \\ = J^2(m) \int d\mathbf{n} \sin^2\vartheta = \frac{d-1}{d} J^2(m), \end{aligned} \tag{5.12}$$

with the coefficient C_d from (3.6). Here and below $\int d\mathbf{n}$ denotes the integral over the d-dimensional sphere, normalized with respect to its area, so that $\int d\mathbf{n}\, 1 = 1$ and $\int d\mathbf{n} \sin^2\vartheta = (d-1)/d$. For the integral (5.11) one has:

$$\begin{aligned} J_2(m) = C_d^2 \int_0^\infty dk \int_0^\infty dq \int d\mathbf{n} \frac{k^d q^d \cos\vartheta \sin^2\vartheta}{k^2+q^2+2kq\cos\vartheta} S(k)S(q) = \\ = 2C_d^2 \int_0^\infty dk \int_0^k dq \int d\mathbf{n} \frac{k^d q^d \cos\vartheta \sin^2\vartheta}{k^2+q^2+2kq\cos\vartheta} S(k)S(q), \end{aligned} \tag{5.13}$$

where we have used the symmetry of the integrand and integration area in k and q.

In order to find the renormalization constant, we need not the entire exact expression (5.13) for the integral $J_2(m)$, we rather need its UV divergent part. The simple power counting shows that the UV divergence of the integral (5.13) is generated by the region in which the both integration momenta k and q are large. Therefore, the integral (5.13) contains only a first-order pole in ε, and the coefficient in $1/\varepsilon$ does not change when the integration area $[0,\infty]$ for the momentum k is restricted from below by some finite limit, for example, $[m,\infty]$. Furthermore, the IR regularization of the integral is then provided by this finite lower limit, and one can simply set $m=0$ in the functions $S(k)$, $S(q)$, which gives:

$$J_2(m) \simeq 2C_d^2 \int_m^\infty dk \int_0^k dq \int d\mathbf{n} \frac{k^{-\varepsilon} q^{-\varepsilon} \cos\vartheta \sin^2\vartheta}{k^2+q^2+2kq\cos\vartheta}. \tag{5.14}$$

Here and below $\simeq$ means the equality up to the terms finite for $\varepsilon \to 0$. From the dimensionality considerations, it is obvious that $J_2(m) = m^{-2\varepsilon} f(\varepsilon)$, where $f(\varepsilon)$ contains a first-order pole in ε. It then follows that

$$J_2(m) = -\frac{1}{2\varepsilon} \mathcal{D}_m J_2(m) \tag{5.15}$$

(we recall the notation $\mathcal{D}_m \equiv m\partial/\partial m$). The representation (5.15) allows one to get rid of the integration over k in (5.14):

$$J_2(m) \simeq C_d^2 \, \frac{m^{-2\varepsilon}}{\varepsilon} \int_0^1 dx \int d\mathbf{n} \, \frac{x^{-\varepsilon} \, \cos\vartheta \, \sin^2\vartheta}{1 + x^2 + 2x \cos\vartheta}, \tag{5.16}$$

where we have performed the substitution $q \equiv mx$. The pole in (5.16) is isolated explicitly, the integral is UV convergent and one can set $\varepsilon = 0$ in the integrand:

$$J_2(m) \simeq C_d^2 \, \frac{m^{-2\varepsilon}}{\varepsilon} \int_0^1 dx \int d\mathbf{n} \, \frac{\cos\vartheta \, \sin^2\vartheta}{1 + x^2 + 2x \cos\vartheta}. \tag{5.17}$$

The integrations in (5.17) are performed explicitly:

$$J_2(m) \simeq C_d^2 \, \frac{m^{-2\varepsilon}}{2\varepsilon} \int d\mathbf{n} \, \vartheta \, \cos\vartheta \, \sin\vartheta = C_d^2 \, \frac{m^{-2\varepsilon}(1-d)}{2\varepsilon d^2}. \tag{5.18}$$

Combining the expressions (5.9), (5.12), and (5.18) we obtain:

$$D_3 = J^2(m) g_0^2 \left[\frac{\alpha^2}{4} + \frac{\alpha(1-\alpha)(d-1)}{8d} \right] + g_0^2 C_d^2 \, \frac{\alpha(1-\alpha)(d-1)\, m^{-2\varepsilon}}{8\varepsilon d^2}. \tag{5.19}$$

Within our accuracy, the renormalization of the expression (5.19) is reduced to the substitution $g_0 \to g\mu^{\varepsilon}$, which gives:

$$D_3 = J^2(m) \mu^{2\varepsilon} \left[\frac{\alpha^2 g^2}{4} + \frac{\alpha g^2 (1-\alpha)(d-1)}{8d} \right] + g^2 C_d^2 \, \frac{\alpha(1-\alpha)(d-1)(\mu/m)^{2\varepsilon}}{8\varepsilon d^2}. \tag{5.20}$$

Now let us turn to the diagram D_4. It is given by the expression

$$\begin{aligned} D_4 = \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \int \frac{d\mathbf{k}}{(2\pi)^d} \int \frac{d\mathbf{q}}{(2\pi)^d} \times \\ \times \frac{R_{ij}(\mathbf{k}) k_i (k+q)_j R_{ps}(\mathbf{k}) q_p q_s S(k) S(q)}{(\mathrm{i}\omega + \nu_0 k^2) |\mathrm{i}\omega' + \nu_0 q^2|^2 (-\mathrm{i}(\omega+\omega') + \nu_0 (\mathbf{k}+\mathbf{q})^2)} = \\ = \frac{\alpha^2 g_0^2}{8} [J^2(m) + J_3(m)], \end{aligned} \tag{5.21}$$

where we have performed the integrations over the frequencies and made use of the symmetry in k and q; the integral $J_3(m)$ is given by:

$$J_3(m) \equiv \int \frac{d\mathbf{k}}{(2\pi)^d} \int \frac{d\mathbf{q}}{(2\pi)^d} \, \frac{\mathbf{k}\cdot\mathbf{q} \, S(k) S(q)}{k^2 + q^2 + \mathbf{k}\cdot\mathbf{q}}. \tag{5.22}$$

Proceeding as for the integral $J_2(m)$ above, we arrive at the expression

$$J_3(m) \simeq C_d^2 \frac{m^{-2\varepsilon}}{\varepsilon} \int_0^1 dx \int d\mathbf{n} \frac{\cos\vartheta}{1+x^2+x\cos\vartheta}, \tag{5.23}$$

which is analogous to the expression (5.17) for $J_2(m)$. In contrast to (5.17), after the integration over x in (5.23) we arrive at the integral over the angles which cannot be calculated explicitly. We rather expand the integrand in (5.23) in $\cos\vartheta$:

$$\begin{aligned} &\int_0^1 dx \int d\mathbf{n} \frac{\cos\vartheta}{1+x^2+x\cos\vartheta} = \\ &= \int_0^1 dx \int d\mathbf{n} \frac{\cos\vartheta}{1+x^2} \sum_{k=0}^{\infty} \left(-\frac{x\cos\vartheta}{1+x^2}\right)^k, \end{aligned} \tag{5.24}$$

and use the formulas

$$\begin{aligned} \int d\mathbf{n}\, \cos^{2k}\vartheta &= \frac{(2k-1)!!}{d(d+2)\dots(d+2k-2)}, \\ \int d\mathbf{n}\, \cos^{2k+1}\vartheta &= 0, \\ \int_0^1 dx \frac{x^{2k+1}}{(1+x^2)^{2k+2}} &= \frac{(k!)^2}{4(2k+1)!}. \end{aligned} \tag{5.25}$$

For the series in (5.24) this gives (we omit an overall minus sign):

$$\frac{1}{4d}\sum_{k=0}^{\infty} \frac{(2k+1)!!(k!)^2}{(2k+1)!(d+2)\dots(d+2k)} = \frac{1}{4d}\sum_{k=0}^{\infty} \frac{k!}{4^k(d/2+1)\dots(d/2+k)} \equiv h(d)/4d, \tag{5.26}$$

with the function $h(d)$ entering into the expressions (3.15)–(3.18).

Combining the expressions (5.21), (5.23), (5.26), and performing the replacement $g_0 \to g\mu^{\varepsilon}$ we obtain

$$D_4 = J^2(m)\mu^{2\varepsilon}\frac{\alpha^2 g^2}{8} - \frac{\alpha^2 g^2 h(d) C_d^2 (\mu/m)^{2\varepsilon}}{32 d\varepsilon}. \tag{5.27}$$

The diagram D_5 is simply given by

$$D_5 = D_1^2 = J^2(m)\mu^{2\varepsilon}\frac{\alpha^2 g^2}{4}, \tag{5.28}$$

and D_6 contains effectively a closed circuit of retarded propagators and vanishes identically. Therefore, the function Γ_n in the two-loop order of the renormalized perturbation theory has the form

$$\Gamma_n = 1 + \frac{n(n-1)}{2}(D_1 + 2D_2 + D_3) + n(n-1)(n-2)D_4 + \frac{n(n-1)(n-2)(n-3)}{8}D_5, \tag{5.29}$$

with the symmetry coefficients from Table II and the explicit expressions for D_i given in (5.6), (5.20), (5.27), and (5.28).

Within our accuracy, the renormalization constant (5.2) has the form

$$Z_n^{-1} = 1 + \frac{a_{11} g}{\varepsilon} + \frac{a_{21} g^2}{\varepsilon} + \frac{a_{22} g^2}{\varepsilon^2} + O(g^3), \tag{5.30}$$

and the requirement that the function (5.1) be UV finite in the first order in g gives:

$$\frac{a_{11} g}{\varepsilon} + \frac{n(n-1)}{2} g D^{(1)} = \text{ UV finite,} \tag{5.31}$$

with the coefficient $D^{(1)}$ defined in (5.6). The expansion in ε of the integral $J(m)$ from (2.8) entering into the expressions for D_i has the form

$$\mu^{\varepsilon} J(m) = \frac{C_d}{\varepsilon} \left[1 + \varepsilon \left(\frac{\psi(1) - \psi(d/2)}{2} + \ln(\mu/m) \right) \right] + O(\varepsilon), \tag{5.32}$$

where $\psi(z) \equiv d \ln \Gamma(z)/dz$. From (5.6), (5.31), and the first term of the expansion (5.32) one obtains:

$$a_{11} = -\alpha n(n-1) C_d/4. \tag{5.33}$$

The UV finiteness of the function (5.1) in the order g^2 implies:

$$\begin{aligned} &\frac{a_{21} g^2}{\varepsilon} + \frac{a_{22} g^2}{\varepsilon^2} + \frac{a_{11} g}{\varepsilon} \, \frac{n(n-1)}{2} g D^{(1)} + \frac{n(n-1)}{2} (g^2 \, D^{(2)} + D_3) + \\ &+ n(n-1)(n-2) D_4 + \frac{n(n-1)(n-2)(n-3)}{8} D_5 = \text{ UV finite,} \end{aligned} \tag{5.34}$$

which along with the expressions (5.6), (5.20), (5.27), (5.28), (5.33) and the expansion (5.32) yields:

$$a_{21}/C_d^2 = \frac{n(n-1)\alpha(\alpha-1)(d-1)}{16 d^2} + \frac{n(n-1)(n-2)\alpha^2 \, h(d)}{32 d}, \tag{5.35a}$$

$$\begin{aligned} a_{22}/C_d^2 = &\frac{\alpha^2 n^2 (n-1)^2}{16} - \frac{\alpha^2 n(n-1)}{8} + \frac{n(n-1)\alpha(\alpha-1)(d-1)}{16 d} - \\ &- \frac{\alpha^2 n(n-1)(n-2)}{8} - \frac{\alpha^2 n(n-1)(n-2)(n-3)}{32} = \\ &= \frac{\alpha \, n(n-1)}{32} [\alpha \, n(n-1) - 2(\alpha + d - 1)/d]. \end{aligned} \tag{5.35b}$$

We note that the $O(1)$ terms of the expansion (5.32) cancel out in the Eq. (5.34) and therefore give no contribution to the coefficients a_{ij}.

For the corresponding anomalous dimension $\gamma_n \equiv \widetilde{\mathcal{D}}_\mu \ln Z_n$ we have:

$$\gamma_n \equiv \widetilde{\mathcal{D}}_\mu \ln Z_n = \beta(g) \partial_g \ln Z_n = [-\varepsilon + \gamma_\nu(g)] \mathcal{D}_g \ln Z_n, \tag{5.36}$$

with the RG functions $\beta(g)$ and $\gamma_\nu(g)$ from (3.5). Within our accuracy (5.36) yields

$$\gamma_n = a_{11}g + 2a_{21}g^2 + \frac{1}{\varepsilon}\left[-a_{11}g\gamma_\nu + 2a_{22}g^2 - (a_{11}g)^2\right], \tag{5.37}$$

and using the explicit expressions (5.33), (5.35) one obtains:

$$\gamma_n = \frac{-\alpha n(n-1)u}{4} + \frac{n(n-1)\alpha(\alpha-1)(d-1)u^2}{8d^2} + \frac{n(n-1)(n-2)\alpha^2 h(d)u^2}{16d} + O(g^3), \tag{5.38}$$

where $u \equiv gC_d$. It follows from the explicit expressions (3.7), (5.33), and (5.35) that the coefficient in $1/\varepsilon$ in the expression (5.37) for γ_n vanishes: $-a_{11}g\gamma_\nu + 2a_{22}g^2 - (a_{11}g)^2 = 0$. This is a manifestation of the general fact that the function γ_n is UV finite, i.e., it has no poles in ε.

Substituting the anomalous dimension (5.38) into the expression (3.10) and performing the replacement $g \to g_*$ with g_* from (3.8), we arrive at the desired expression (3.15) for the critical dimension of the composite operator θ^n.

It is worth noting that the case $d = 1$ is exceptional in the sense that "there are no angles in one dimension." We have performed all the calculation directly in $d = 1$ and checked that the one-dimensional exponents are indeed obtained from the general expressions like (3.19) by the substitution $d = 1$.

VI. DISCUSSION AND CONCLUSION

We have applied the RG and OPE methods to the simple model (1.1), (1.2), (1.8), which describes the advection of a passive scalar by the non-solenoidal ("compressible") velocity field, decorrelated in time and self-similar in space. We have shown that the correlation functions of the scalar field in the convective range exhibit anomalous scaling behavior; the corresponding anomalous exponents have been calculated to the second order of the ε expansion (the two-loop approximation), see (3.15)–(3.20). They depend on a free parameter, the ratio $\alpha = D_0'/D_0$ of the amplitudes in the transversal and longitudinal parts of the velocity correlator, and are in this sense non-universal. In the language of the RG, the non-universality of the exponents is related to the fact that the fixed point of the RG equations is degenerated: its coordinate depends continuously on α.

In contrast to the model (1.3), where the anomalous exponents are determined by the critical dimensions of the composite operators $(\partial_i\theta\partial_i\theta)^n$, the exponents in the model (1.8) are related to the critical dimensions of the monomials θ^n, the powers of the field itself, and these dimensions appear to be nonlinear functions of n. This explains the important difference between the anomalous scaling behavior of the model (1.3) and that of the model (1.8): in the latter, the correlation functions in the convective range depend substantially on both the IR and UV characteristic scales, and the structure functions are independent of the separation $r = |\mathbf{x} - \mathbf{x}'|$. The monomials θ^n in the model (1.8) also provide an example of the power field operators *without derivatives*, whose correlation functions exhibit multifractal behavior (another interesting example is the field theoretical model of a growth process considered in [42]). Analogous behavior is demonstrated by the model of a magnetic field, advected passively by the incompressible Gaussian velocity; the corresponding anomalous exponents are calculated to the order ε (ε^2 for the pair correlator).

The anomalous exponent for the pair correlation function has been found exactly for all $0 < \varepsilon < 2$. Its expansion in ε coincides with the result obtained using the RG for all values of the space dimensionality d and ratio α. The agreement between the exact exponent for the pair correlation function and the first two terms of the corresponding ε expansion is also established for a passively advected magnetic field.

These facts support strongly the applicability of the RG technique and the ε expansion to the problem of anomalous scaling for the finite values of ε, at least for low-order correlation functions.

We note that the series in ε for all known exact exponents in the rapid-change models have finite radii of convergence, a rare thing for field theoretical models. In the language of the field theory, this is related to the fact that in the rapid-change models, there is no factorial growth of the number of diagrams in higher orders of the perturbation theory (a great deal of diagrams indeed vanish owing to retardation, see discussion in Sec. II). In its turn, this fact suggests that the series in ε for the unknown exponents (for example, the anomalous exponents in the original Obukhov–Kraichnan model) can also be convergent.

It should also be noted that the asymptotic expressions (1.9), (1.11) result from the fact that the critical dimensions Δ_n are negative, and that the modulus $|\Delta_n|$ increases monotonously with n. This is obviously so within the ε expansion, in which the sign and the n dependence of the dimensions are determined by the first-order terms (1.10), (4.6), while the higher-order terms are treated as small corrections. However, for finite values of ε the higher-order terms can, in principle, change these features of the dimensions. Indeed, the n^3 contribution in the second-order approximation for Δ_n is positive, see e.g. (3.15), so that Δ_n also becomes positive, provided n is large enough. Of course, this conclusion is based on the second-order approximation of the ε expansion and is therefore not definitive: the higher-order terms of the ε expansion contain additional powers of n, so that the actual expansion parameter appears to be εn rather than ε itself, cf. [22,42]. Therefore, the correct analysis of the large n behavior of the anomalous exponents requires resummation of the ε expansions with the additional condition that $\varepsilon n \simeq 1$. This is clearly not a simple problem and it requires considerable improvement of the existing technique.

ACKNOWLEDGMENTS

The authors are thankful to A. N. Vasil'ev for clarifying discussions and to A. V. Runov for his help in preparing the Figures. One of the authors (N.V.A.) is thankful to G. L. Eyink and R. H. Kraichnan for interesting remarks regarding the paper [22].

The work was supported by the Russian Foundation for Fundamental Research (grant N 96-02-17-033) and by the Grant Center for Natural Sciences of the Russian State Committee for Higher Education (grant N 97-0-14.1-30).

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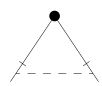
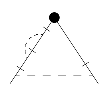
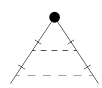
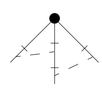
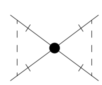
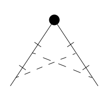
TABLES

TABLE I. Canonical dimensions of the fields and parameters in the model (2.1).

F	θ	θ'	$\mathbf{v}$	ν, ν_0	D_0, D_0'	m, M, μ, Λ	g_0	g, α
d_F^k	0	d	-1	-2	-2+ε	1	ε	0
d_F^ω	-1/2	1/2	1	1	1	0	0	0
d_F	-1	d+1	1	0	ε	1	ε	0

TABLE II. The diagrams of the 1-irreducible Green function $\langle \theta^n(x)\theta(x_1)\ldots\theta(x_2)\rangle$ in the two-loop approximation.

	Diagram	Symmetry coefficient
D_1		$n(n-1)/2$
D_2		$n(n-1)$
D_3		$n(n-1)/2$
D_4		$n(n-1)(n-2)$
D_5		$n(n-1)(n-2)(n-3)/8$
D_6		—